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Classical Euclidean Wormhole Solution and Wave Function for a Nonlinear Scalar Field

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In this paper we consider the classical Euclidean wormhole solution of the Born–Infeld scalar field. The corresponding classical Euclidean wormhole solution can be obtained analytically for both very small and large $\dot{\varphi}$. At the extreme limit of small $\dot{\varphi}$ the wormhole solution has the same format as one obtained by Giddings and Strominger (*Nuclear Physics B* **306**, 890, 1988). At the extreme limit of large $\dot{\varphi}$ the wormhole solution is a new one. The wormhole wave functions can also be obtained for both very small and large $\dot{\varphi}$. These wormhole wave functions are regarded as solutions of quantum-mechanical Wheeler–Dewitt equation with certain boundary conditions.

KEY WORDS: Euclidean wormhole; Born–Infeld field; wormhole wave function; Wheeler–Dewitt equation.

1. INTRODUCTION

Euclidean wormholes in quantum gravity are possibly useful in understanding black hole evaporation; in allowing nonlocal connections that could determine fundamental constants, e.g., Λ (Coleman, 1988), and even as an alternative to the Higgs mechanism (Mignemi and Moss, 1993).

In Einstein theory, the complex scalar field and conformal scalar field have classical Euclidean wormhole solutions. These scalar fields are constrained to be a linear field and the wormholes as solution of quantum-mechanical Wheeler–Dewitt equation has been obtained (Howking and Pape, 1990).

In this paper we consider the classical Euclidean wormhole solution and wormhole wave function with a nonlinear Born–Infeld field.

The corresponding Lagrangian of Born–Infeld field has been first proposed by Heisenberg (1952) to describe the process of meson multiple production connected with strong field regime, as a generalization of the Born–Infeld one,

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$L^{\mathrm{BI}} = b^2 \lfloor \sqrt{1 - (1/2b^2) F_{ik} F^{ik}} - 1 \rfloor$ (Born and Infeld. 1934), that removes the point-charge singularity that mars classical electrodynamics. When the parameter of the field approaches to zero, the corresponding Lagrangian will reduce to linear case. Static and spherically symmetric solutions of Born–Infeld scalar field and corresponding black hole have been recently investigated qualitatively by de Oliveiry (1995). Born–Infeld-type Lagrangians have also been considered in the theory of strings and branes (Boillat and Strumia, 1998; Deser and Gibbons, 1998; de Oliveira, 1995; Feigenbaum, 1998; Palatnik, 1999; Tseytlin, 1986). It was shown that the low energy effective field theory on D-branes is of Born–Infeld type (Tseytlin, 1986). The consistency of the σ-model for the world sheet of string is shown to require that the brane be described by Born–Infeld action, just like in the general curved background requiring consistency of string theory leads to the Einstein–Hilbert action.

The present paper is organized as follows. In Section 2 we obtain the Euclidean wormhole solution of Born–Infeld scalar field. According to the Euler–Lagrangian equation of motion of Born–Infeld scalar field, we can obtain $\dot{\varphi}$ at the limit of large and small cosmological scalar factors R respectively. At such limit condition, we found Euclidean wormholes. In Section 3 we found wormhole wave function of our nonlinear scalar field model. In last section we discuss our results.

2. CLASSICAL WORMHOLE SOLUTION

The Euclidean action of gravitational field interacting with a Born–Infeld-type scalar field is given by

$$S_{\mathrm{E}} = \int \frac{R_{\mathrm{c}}}{16\pi G} \sqrt{g}\, d^4x + \int L_{\mathrm{s}} \sqrt{g}\, d^4x \tag{1}$$

where we have chosen unit so that $c = 1$, R_{c} is the Ricci scalar curvature and the Lagrangian L_{s} of the nonlinear Born–Infeld scalar field is (Born and Infeld, 1934; Heisenberg, 1952)

$$L_{\mathrm{s}} = \frac{1}{\lambda} \left[1 - \sqrt{1 - \lambda \varphi_{,\mu} \varphi_{,\nu} g^{\mu\nu}} \right] \tag{2}$$

When $\lambda \to 0$, based on Taylor expansion (2) approximates to

$$\lim_{\lambda \to 0} L_{\mathrm{s}} = \frac{1}{2} \varphi_{,\mu} \varphi_{,\nu} g^{\mu\nu} \tag{3}$$

We choose the standard Euclidean and closed R-W metric

$$ds^2 = d\tau^2 + R^2(\tau) \left\{ \frac{dr^2}{1 - r^2} + r^2 [(d\theta^2) + \sin^2\theta (d\varphi^2)] \right\} \tag{4}$$

Where τ is the Euclidean radial coordinate and $R(\tau)$ is the radius of curvature of a 3D sphere. According to the "cosmological principle," R must only depend on

τ. We write Einstein equations as

$$-\frac{3\dot{R}^2}{R^2} + \frac{3}{R^2} = 8\pi G T_0^0 \tag{5}$$

$$-\frac{2\ddot{R}}{R} - \frac{\dot{R}}{R^2} + \frac{1}{R^2} = 8\pi G T_1^1 = 8\pi G T_2^2 = 8\pi G T_3^3 \tag{6}$$

where the upper index "•" denotes the derivative with respect to τ. We substitute the Lagrangian (2) into Eule–Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial L_s}{\partial \dot{\varphi}}\right) - \frac{\partial L_s}{\partial \varphi} = 0 \tag{7}$$

Then we obtain

$$\frac{R^6 \dot{\varphi}^2}{1 + \lambda \dot{\varphi}^2} = W_0 \tag{8}$$

and consequently

$$\dot{\varphi} = \sqrt{\frac{W_0}{R^6 - W_0 \lambda}} \tag{9}$$

Where W_0 is a constant of integration. We write components of energy–momentum tensor of Born–Infeld scalar field as

$$T^\mu \nu = \frac{g^{\mu\rho} \varphi_{,\rho} \varphi_{,\nu}}{\sqrt{1 - \lambda \varphi_{,\mu} \varphi_{,\nu} g^{\mu\nu}}} - \delta_\nu^\mu L_s \tag{10}$$

Substituting Eqs. (9) and (2) into (10), we obtain

$$T_0^0 = \frac{1}{\lambda}\left[\sqrt{R^6 - \lambda W_0}/R^3 - 1\right] \tag{11}$$

$$T_1^1 = T_2^2 = T_3^3 = -\frac{1}{\lambda}\left[1 - R^3/\sqrt{R^6 - \lambda W_0}\right] \tag{12}$$

Substituting (11) into Einstein equation (5), we can obtain

$$-\frac{3\dot{R}^2}{R^2} + \frac{3}{R^2} = \frac{8\pi G}{\lambda}\left[\frac{\sqrt{R^6 - \lambda W_0}}{R^3} - 1\right] \tag{13}$$

$$\dot{R}^2 = 1 - \frac{8\pi G}{3\lambda}\left[R^2\sqrt{1 - \lambda W_0 R^{-6}} - R^2\right] \tag{14}$$

From Eq. (9), we find that R is very small or very large when $\dot{\varphi}$ is very large or very small respectively. Assuming that $\dot{\varphi}$ is very small (i.e., R is very large), Eq. (14) becomes

$$\dot{R}^2 = 1 + \frac{4\pi G W_0}{3R^4} \tag{15}$$

When $W_0 < 0$, the wormhole solution of Eq. (15) is

$$\frac{\tau}{R_0} = \sqrt{\frac{1}{2}} F\left[\cos^{-1}\left(\frac{R_0}{R}\right), \sqrt{\frac{1}{2}}\right] - \sqrt{2} E\left[\cos^{-1}\left(\frac{R_0}{R}\right), \sqrt{\frac{1}{2}}\right] + \frac{\sqrt{R^4 - R_0^4}}{R_0 R} \tag{16}$$

Where $R_0 = \sqrt[4]{\frac{-4\pi G W_0}{3}}$. We note that this wormhole solution has the same format as one obtained by Giddings and Strominger (1988; Palatnik, 1998). When $\dot{\varphi}$ is very large (i.e., R is very small), we can obtain from Eq. (14)

$$\dot{R}^2 = 1 - \frac{8\pi G\sqrt{\frac{-W_0}{9\lambda}}}{R} \tag{17}$$

We restrict $\lambda > 0$, integrating (17) we can obtain wormhole solution of Eq. (17), that is

$$R\sqrt{1 - \frac{N}{R}} + N \log\left[\frac{\sqrt{\frac{R}{N}} - 1 + \sqrt{\frac{R}{N}} - 1}{\sqrt{\frac{R}{N}} - 1 - \sqrt{\frac{R}{N}} + 1}\right] = \tau \tag{18}$$

Where $N = 8\pi G\sqrt{\frac{-W_0}{9\lambda}} > 0$. From Eq. (18) we can find that $\lim_{\tau\to\infty} R(\tau) = \infty$. Using $\dot{R}(0) = 0$ from Eq. (17) we can obtain the size of wormhole throat: $R(0) = N$ and $\ddot{R}(0) = \frac{1}{2N} > 0$. Thus we obtain a new wormhole solution (18).

3. WORMHOLE WAVE FUNCTION

It is possible that the wormholes are regarded as solutions of quantum-mechanical Wheeler–Dewitt (WD) equation (Coule, 1992; Gonzalez, 1990; Hawking and Pape, 1990). These wave functions have to obey certain boundary conditions in order that they represent wormholes. The wave function will be damped at large radius R, i.e., such a wave function tends to zero as $R \to \infty$, and when R nears 0, it should be oscillatory (Palatnik, 2001). Wave function should tend to a constant as $R \to 0$ (Coule, 1997). The Lorentz action of the gravitational field interacting with a Born–Infeld-type scalar field is given by

$$S = \int \frac{R_c}{16\pi G}\sqrt{-g}\, d^4x + \int L_s\sqrt{-g}\, d^4x \tag{19}$$

where R_c is the Ricci scalar curvature and L_s is Eq. (2). However, in Eq. (19), $g_{\mu\nu}$ is decided by Eq. (20). The closed R–W space-time metric is

$$ds^2 = -dt^2 + R^2(t)\left\{\frac{dr^2}{1 - r^2} + r^2[d\theta^2 + \sin^2\theta(d\varphi^2)]\right\} \tag{20}$$

Using Eq. (20) and integrating space components, the action (19) becomes (The upper dot means the derivative with respect to the time t.):

$$S = \int \frac{3\pi}{4G}(1-\dot{R}^2)R\,dt + \int 2\pi^2 R^3 \left[\frac{1}{\lambda}\left(1-\sqrt{1-\lambda\varphi_{,\mu}\varphi_{,\nu}g^{\mu\nu}}\right)\right]dt$$
$$\equiv \int L_{\mathrm{g}}\,dt + \int L_{\mathrm{s}}\,dt \tag{21}$$

To quantize the model, we first find the canonical moment

$$P_{\mathrm{R}} = \left(\frac{\partial L_{\mathrm{g}}}{\partial \dot{R}}\right) = -\left(\frac{3\pi}{2G}\right)R\dot{R}, \qquad P_{\varphi} = \left(\frac{\partial L_{\mathrm{s}}}{\partial\dot{\varphi}}\right) = \left(2\pi^2 R^3\dot{\varphi}/\sqrt{1+\lambda\dot{\varphi}^2}\right)$$

and the Hamiltonian $H = P_{\mathrm{R}}\dot{R} + P_{\varphi}\dot{\varphi} - L_{\mathrm{g}} - L_{\mathrm{s}}$,

$$H = -\frac{G}{3\pi R}P_R^2 - \frac{3\pi}{4G}R + \frac{2\pi^2 R^3}{\lambda}\left(1-\sqrt{1-\frac{\lambda P_{\varphi}^2}{4\pi^4 R^6}}\right) \tag{22}$$

For small $\dot{\varphi}$, the Hamiltonian (22) can be simplified by using the Taylor expansion

$$H = -\frac{G}{3\pi R}P_R^2 - \frac{3\pi}{4G}R + \frac{P_{\varphi}^2}{4\pi^2 R^3} - \frac{\lambda P_{\varphi}^4}{4\pi^4 R^9} \tag{23}$$

If $\dot{\varphi}$ is large, then Eq. (22) becomes

$$H = -\frac{G}{3\pi R}P_R^2 - \frac{3\pi}{4G}R + \frac{2\pi^2 R^3}{\lambda} \tag{24}$$

The WD equation is obtained from $\hat{H}\psi = 0$ and Eqs. (23) and (24) by replacing $P_{\mathrm{R}} \to -i(\frac{\partial}{\partial R})$ and $P_{\varphi} \to -i(\frac{\partial}{\partial\varphi})$. Then we obtain

$$\left[\frac{\partial^2}{\partial R^2} + \frac{P}{R}\frac{\partial}{\partial R} - \frac{1}{R^2}\frac{\partial}{\partial\Phi^2} - \frac{\lambda}{16\pi^4 R^8}\frac{\partial^4}{\partial\Phi^4} - U(R)\right]\psi = 0 \tag{25}$$

and

$$\left[\frac{\partial^2}{\partial R^2} + \frac{P}{R}\frac{\partial}{\partial R} - u(R)\right]\psi = 0 \tag{26}$$

where $\Phi^2 = 4\pi G\varphi^2/3$ and the parameter P represents the ambiguity in the ordering of factors R and $\frac{\partial}{\partial R}$ in the first term of Eqs. (23) and (24). We have also denoted

$$U(R) = \left(\frac{3\pi}{2G}\right)^2 R^2$$
$$u(R) = \left(\frac{3\pi}{2G}\right)^2 R^2\left[1-\frac{8\pi G}{3\lambda}R^2\right]$$

Equations (25) and (26) are the WD equations corresponding to action (19) in the cases of small and large $\dot{\varphi}$ respectively. Together we can obtain the equation of motion of Born–Infeld scalar field when we substitute the Lagrangian L_s into the Eule-Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial L_s}{\partial \dot{\varphi}}\right) - \frac{\partial L_s}{\partial \varphi} = 0 \tag{27}$$

Then we obtain

$$\dot{\varphi} = \sqrt{\frac{C}{R^6 + C\lambda}} \tag{28}$$

The upper dot means the derivative with respect to the t. Where C is a constant of integration. From Eq. (28) we find that R is very small or very large when $\dot{\varphi}$ is very large or very small respectively. In other word, Eqs. (25) and (26) are the WD equations corresponding to action (19) in the cases of large and small R respectively. When R is very large, we take the ambiguity of ordering factor $P = -1$ and set transformation $(R/R_0)^2 = \sigma$, with R_0 the Planck's length. Choosing appropriate units makes the Planck constant $\hbar = 1$, speed of light $c = 1$, and $R_0\sqrt{\frac{4G}{3\pi}} \approx 10^{-33}$ cm. Then Eq. (25) becomes

$$\frac{\partial^2 \psi}{\partial \sigma^2} - \frac{1}{\sigma^2}\frac{\partial^2 \psi}{\partial \Phi^2} - \frac{\lambda}{16\pi^4 R_0^6 \sigma^5}\frac{\partial^4 \psi}{\partial \Phi^4} - \tilde{U}\psi = 0 \tag{29}$$

where $\tilde{U} = (3\pi/4G)^2 R_0^4$. Assuming $\psi(\sigma, \Phi) = Q(\sigma)e^{-k\Phi}$, with K an arbitrary constant, Eq. (29) takes the form:

$$\frac{d^2 Q}{d\sigma^2} - \left(\frac{K^2}{\sigma^2} + \frac{\mu K^4}{\sigma^5} + \tilde{U}\right) Q = 0 \tag{30}$$

Where $\mu = \lambda/16\pi^4 R_0^6$. When R (and consequently σ) is very large, Eq. (30) approximates to

$$\frac{d^2 Q}{d\sigma^2} - \beta^2 Q = 0 \tag{31}$$

Where $\beta = (\frac{3\pi}{4G})R_0^2$. The solution of Eq. (31) is

$$Q = \exp(-\beta\sigma) \tag{32}$$

From (32) we can find that wave function $\psi \to 0$ when $R \to \infty$ (and consequently $\sigma \to \infty$).

If R is very small, we take the ambiguity of ordering factor $P = -1$ and set the transformation $(R/R_0)^2 = \sigma$, with R_0 the Planck length. Choosing appropriate units makes the Planck constant $\hbar = 1$, the speed of light $c = 1$ and

$R_0 = \sqrt{\frac{4G}{3\pi}} \approx\ 10^{-33}$ cm. Then Eq. (26) becomes

$$\frac{d^2\psi}{d\sigma^2} - \left(\frac{3\pi}{4G}\right)^2 R_0^4 \left(1 - \frac{8\pi G}{\lambda}\sigma R_0^2\right)\psi = 0 \tag{33}$$

When $R \gg \sqrt{\frac{3\lambda}{8\pi G}}$ (and consequently $\sigma \gg \frac{3\lambda}{8\pi G R_0^2}$), Eq. (33) can be approximated as

$$\frac{d^2\psi}{d\sigma^2} - \gamma^2\sigma\psi = 0 \tag{34}$$

Where $\gamma = (\frac{3\pi^3}{2G\lambda})^{1/2}\ R_0^2$. Equation (34) has the solution

$$\psi = \sqrt{\sigma} Z_{\frac{1}{3}}\left(\frac{2\gamma}{3}\sigma^{3/2}\right) \tag{35}$$

The solution (35) shows that the wave function oscillates when R nears zero radius. When $R \ll \sqrt{\frac{3\lambda}{8\pi G}}$ Eq. (33) can be approximated as

$$\frac{d^2\psi}{d\sigma^2} - \left(\frac{3\pi}{4G}\right)^2 R_0^4\psi = 0 \tag{36}$$

Equation (36) has the solution

$$\psi = Ne^{\frac{-3\pi}{4G}R^2} \tag{37}$$

This wave function tends to a constant as R tends to a zero. In the geometry described by the R–W metric, the probability of wormhole situated between $R \rightarrow R + dR$ is

$$\omega(R) \propto \psi^2 R^2\, dR \tag{38}$$

The probability density is $\psi^2 R^2$. The position of the maximum probability can be determined by

$$\frac{d}{dR}(\psi^2 R^2) = 0 \tag{39}$$

From (39) we can obtain

$$R = \sqrt{\frac{4G}{6\pi}} \tag{40}$$

Equation (40) implies that most probable radius of wormhole is of the Planck scale, namely the quantum effect can make a wormhole survive gravitational collapse.

4. CONCLUSION

At the extreme limits of small $\dot{\varphi}$, the classical wormhole solution of the Born–Infeld scalar field has the same format as one obtained by Giddings and Strominger. If $\dot{\varphi}$ is very large, a new wormhole solution can be obtained. From the Eule–Lagrange equation of the Born–Infeld scalar field, we find that cosmological scale factors are very large or very small when $\dot{\varphi}$ is very small or very large respectively. We obtain the wormhole wave function. It is the solution of quantum-mechanical Wheeler–Dewitt equation with certain boundary conditions. The wave function is exponentially damped for large three geometries and the wave function tends to a zero as cosmological scale factors tend to an infinity. They oscillate near zero radius; it tends to a constant as cosmological radius tends to a zero.

In quantum cosmology with a Born–Infeld-type scalar field, the wave function of Universe is obtained. An inflationary Universe is predicted with the largest possible vacuum energy (Lu *et al.*, 1999).

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